HEAT AND MOISTURE TRANSFER OF SOME BODIES WITH BOUNDARY CONDITIONS OF THE SECOND KIND

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Inzhenerno-Fizicheskii Zhurnal, Vol. 11, No. 6, pp. 740–746, 1966

UDC 66.047

This paper gives a general solution of the second boundary-value problem of heat and moisture transfer for bodies of the type of an infinite wall and infinite cylinder on the basis of Luikov's generalized system of equations [1].

The main equations and boundary conditions of the problem are formulated in the following way.

The system of differential equations,

$$\frac{\partial T}{\partial \tau} = a\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{i}{\xi}\frac{\partial T}{\partial \xi}\right) + \frac{\varepsilon\rho}{c}\frac{\partial U}{\partial \tau},$$

$$\tau_{rm}\frac{\partial^2 U}{\partial \tau^2} + \frac{\partial U}{\partial \tau} = a_m\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{i}{\xi}\frac{\partial U}{\partial \xi}\right) + a_m\delta\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{i}{\xi}\frac{\partial T}{\partial \xi}\right),$$

$$R_1 < \xi < R_2, \quad i = 0, 1. \tag{1}$$

The initial conditions,

$$T(\xi,\ 0) = \Theta(\xi),$$

$$U(\xi,\ 0) = \varphi(\xi),$$

$$\frac{\partial U(\xi,\ 0)}{\partial \xi} = \psi(\xi). \tag{2}$$

The boundary conditions,

$$\frac{\partial T(R_j,\ \tau)}{\partial \xi} = (-1)^j q_j(\tau),$$

$$\frac{\partial U(R_j,\ \tau)}{\partial \xi} + \delta\frac{\partial T(R_j,\ \tau)}{\partial \xi} = (-1)^{j+1}\frac{m_j(\tau)}{a_m\gamma_0},$$

$$j = 1,\ 2. \tag{3}$$

To solve the problem we use the finite integral transform [2]

$$\bar{f} = \int_{R_1}^{R_2} \xi^i f(\xi) W_0\left(\omega_n \frac{\xi}{L}\right) d\xi \tag{4}$$

with the inversion formula

$$f(\xi) = \frac{\int_{R_1}^{R_2} \xi^i f(\xi)\, d\xi}{\int_{R_1}^{R_2} \xi^i\, d\xi} + \sum_{n=1}^{\infty} \frac{\bar{f} W_0(\omega_n \xi/L)}{\int_{R_1}^{R_2} \xi^i W_1^2(\omega_n \xi/L)\, d\xi}. \tag{5}$$

We have a system of ordinary differential equations

$$\frac{d\bar{T}}{d\tau} - \frac{\varepsilon\rho}{c}\frac{d\bar{U}}{d\tau} + \frac{\mu_n^2}{\mathrm{Lu}}\bar{T} = \Phi_1(\tau),$$

$$\tau_{rm}\frac{d^2\bar{U}}{d\tau^2} + \frac{d\bar{U}}{d\tau} + \mu_n^2\bar{U} + \delta\mu_n^2\bar{T} = -\Phi_2(\tau) \tag{6}$$

with initial conditions

$$\bar{T}(0) = \int_{R_1}^{R_2} \xi^i \Theta(\xi) W_0\left(\omega_n \frac{\xi}{L}\right) d\xi = \bar{\Theta},$$

$$\bar{U}(0) = \int_{R_1}^{R_2} \xi^i \varphi(\xi) W_0\left(\omega_n \frac{\xi}{L}\right) d\xi = \bar{\varphi},$$

$$\frac{d\bar{U}(0)}{d\tau} = \int_{R_1}^{R_2} \xi^i \psi(\xi) W_0\left(\omega_n \frac{\xi}{L}\right) d\xi = \bar{\psi}, \tag{7}$$

where

$$\Phi_1(\tau) = \frac{1}{c\gamma_0}\left[R_2^i W_0\left(\omega_n \frac{R_2}{L}\right) q_2(\tau) + R_1^i W_0\left(\omega_n \frac{R_1}{L}\right) q_1(\tau)\right],$$

$$\Phi_2(\tau) = \frac{1}{\gamma_0}\left[R_2^i W_0\left(\omega_n \frac{R_2}{L}\right) m_2(\tau) + R_1^i W_0\left(\omega_n \frac{R_1}{L}\right) m_1(\tau)\right].$$

We apply the Laplace transform to (6) and (7)

$$\tilde{v} = \int_0^\infty v \exp(-s\tau)\, d\tau.$$

We obtain

$$\left(s + \frac{\mu_n^2}{\mathrm{Lu}}\right)\tilde{\bar{T}} - \frac{\varepsilon\rho}{c} s\tilde{\bar{U}} = \tilde{\Phi}_1 + \bar{\Theta} - \frac{\varepsilon\rho}{c}\bar{\varphi},$$

$$\delta\mu_n^2\tilde{\bar{T}} + (\tau_{rm}s^2 + s + \mu_n^2)\tilde{\bar{U}} = -\tilde{\Phi}_2 + \tau_{rm}(s+1)\bar{\varphi} + \tau_{rm}\bar{\psi}.$$

Hence

$$\tilde{\bar{T}} = \tau_{rm}(\tilde{\Phi}_1 + \bar{\Theta})\frac{s^2}{\Delta(s)} + \left[\tilde{\Phi}_1 - \frac{\varepsilon\rho}{c}\tilde{\Phi}_2 + \bar{\Theta} - \frac{\varepsilon\rho}{c}(1-\tau_{rm})\bar{\varphi} + \frac{\varepsilon\rho}{c}\tau_{rm}\bar{\psi}\right]\frac{s}{\Delta(s)} + \mu_n^2\left(\tilde{\Phi}_1 + \bar{\Theta} - \frac{\varepsilon\rho}{c}\bar{\varphi}\right)\frac{1}{\Delta(s)},$$

$$\tilde{\bar{U}} = \frac{\varepsilon\rho}{c}\tau_{rm}\bar{\varphi}\frac{s^2}{\Delta(s)} - \left[\tilde{\Phi}_2 - \tau_{rm}\left(\frac{\varepsilon\rho}{c}\frac{\mu_n^2}{\mathrm{Lu}} + 1\right)\bar{\varphi} - \tau_{rm}\bar{\psi}\right]\frac{s}{\Delta(s)} - \mu_n^2\left[\frac{1}{\mathrm{Lu}}\tilde{\Phi}_2 - \delta\tilde{\Phi}_1 + \left(\mathrm{Fe} - \tau_{rm}\frac{1}{\mathrm{Lu}}\right)\bar{\varphi} - \frac{1}{\mathrm{Lu}}\tau_{rm}\bar{\psi} - \delta\bar{\Theta}\right]\frac{1}{\Delta(s)},$$

where

$$\Delta(s)=\tau_{rm}s^3+\left(1+\tau_{rm}\frac{\mu_n^2}{\mathrm{Lu}}\right)s^2+$$
$$+\mu_n^2\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right)s+\frac{\mu_n^4}{\mathrm{Lu}}.$$

By standard methods [3] we find the originals $\overline{T}$ and $\overline{U}$,

$$\overline{T}=\sum_{\nu=1}^{3}\frac{\exp(s_\nu\tau)}{\Delta'(s_\nu)}\Big\{(\tau_{rm}s_\nu^2+s_\nu+\mu_n^2)\overline{\Theta}-$$
$$-\frac{\varepsilon\rho}{c}[(1-\tau_{rm})s_\nu+\mu_n^2]\overline{\varphi}+$$
$$+\tau_{rm}\frac{\varepsilon\rho}{c}\overline{\psi}+\frac{1}{c\gamma_0}(\tau_{rm}s_\nu^2+s_\nu+\mu_n^2)F(\tau,q)-$$
$$-\frac{\varepsilon\rho}{c\gamma_0}s_\nu F(\tau,m)\Big\},$$

$$\overline{U}=\sum_{\nu=1}^{3}\frac{\exp(s_\nu\tau)}{\Delta'(s_\nu)}\Big\{\mu_n^2\delta\overline{\Theta}+\Big[\tau_{rm}\frac{\varepsilon\rho}{c}s_\nu^2+$$
$$+\tau_{rm}\left(1+\frac{\varepsilon\rho}{c}\frac{\mu_n^2}{\mathrm{Lu}}\right)s_\nu-$$
$$-\mu_n^2\left(\mathrm{Fe}-\tau_{rm}\frac{1}{\mathrm{Lu}}\right)\Big]\overline{\varphi}+\tau_{rm}\left(s_\nu+\frac{\mu_n^2}{\mathrm{Lu}}\right)\overline{\psi}+$$
$$+\mu_n^2\frac{\delta}{c\gamma_0}F(\tau,q)-\frac{1}{\gamma_0}\left(s_\nu+\frac{\mu_n^2}{\mathrm{Lu}}\right)F(\tau,m)\Big\},$$

where s_ν is the root of the cubic equation

$$\Delta(s)=0,$$
$$\Delta'(s)=3\tau_{rm}s^2+2s\left(1+\tau_{rm}\frac{\mu_n^2}{\mathrm{Lu}}\right)+$$
$$+\mu_n^2\left(1+\mathrm{Fe}+\frac{1}{\mathrm{Lu}}\right),$$
$$F(\tau,l)=\int_0^\tau\Big[R_2^iW_0\left(\omega_n\frac{R_2}{L}\right)l_2(t)+$$
$$+R_1^iW_0\left(\omega_n\frac{R_1}{L}\right)l_1(t)\Big]\exp(-s_\nu t)\,dt. \quad (8)$$

We determine the roots of the cubic equation (8). We rewrite this equation as

$$as^3+bs^2+cs+d=0.$$

Here

$$a=\tau_{rm},\quad b=1+\tau_{rm}\frac{\mu_n^2}{\mathrm{Lu}},$$
$$c=\mu_n^2(1+\mathrm{Fe}+1/\mathrm{Lu}),\quad d=\mu_n^4/\mathrm{Lu}.$$

We introduce

$$2g=2b^3/27a^3-bc/3a^2+d/a,$$
$$3h=(3ac-b^2)/3a^2.$$

We can infer that for several structural materials, such as concrete, $g>0$, $h<0$, and the discriminant $D=g^2+h^3<0$.

Then, as we know from [4],

$$s_1=-2r\cos\frac{\varphi}{3}-\frac{b}{3a},$$
$$s_2=+2r\cos(60^\circ-\varphi/3)-b/3a,$$
$$s_3=+2r\cos(60^\circ+\varphi/3)-b/3a, \quad (9)$$

where $\cos\varphi=g/r^3$; $r=\pm\sqrt{|h|}$; the sign of r is the same as the sign of g.

Obviously

$$\cos(60^\circ+\varphi/3)\leqslant 1/2,\quad r=\sqrt{|c/3a-b^2/9a^2|}<b/3a.$$

Hence

$$s_1<0.$$

But [4]

$$s_1\cdot s_2\cdot s_3=-d/a<0.$$

Whence

$$s_2<0.$$

Formulas (9) with due regard to the temporarily introduced symbols give three real different and negative roots s_ν of the third-degree equation (8).

Thus, the final solution of the problem posed has the form

$$T=\frac{2}{(R_2^i+R_1^i)(R_2-R_1)}\int_{R_1}^{R_2}\xi^i\Theta(\xi)\,d\xi+$$
$$+\frac{2}{c\gamma_0(R_2^i+R_1^i)(R_2-R_1)}\int_0^\tau[R_2^iq_2(t)+R_1^iq_1(t)]\,dt-$$
$$-\frac{2\varepsilon\rho}{c\gamma_0(R_2^i+R_1^i)(R_2-R_1)}\int_0^\tau[R_2^im_2(t)+R_1^im_1(t)]\,dt+$$
$$+\sum_{n=1}^{\infty}\frac{W_0\left(\omega_n\frac{\xi}{L}\right)}{\int_{R_1}^{R_2}\xi^iW_1^2\left(\omega_n\frac{\xi}{L}\right)d\xi}\times$$
$$\times\sum_{\nu=1}^{3}\frac{\exp(s_\nu\tau)}{\Delta'(s_\nu)}\Big\{(\tau_{rm}s_\nu^2+s_\nu+\mu_n^2)\overline{\Theta}-$$
$$-\frac{\varepsilon\rho}{c}[(1-\tau_{rm})s_\nu+\mu_n^2]\overline{\varphi}+\tau_{rm}\frac{\varepsilon\rho}{c}\overline{\psi}+$$
$$+\frac{1}{c\gamma_0}(\tau_{rm}s_\nu^2+s_\nu+\mu_n^2)F(\tau,q)-\frac{\varepsilon\rho}{c\gamma_0}s_\nu F(\tau,m)\Big\},$$

$$U=\frac{2}{(R_2^i+R_1^i)(R_2-R_1)}\int_{R_1}^{R_2}\xi^i\varphi(\xi)\,d\xi-$$
$$-\frac{2}{\gamma_0(R_2^i+R_1^i)(R_2-R_1)}\int_0^\tau[R_2^im_2(t)+R_1^im_1(t)]\,dt+$$
$$+\sum_{n=1}^{\infty}\frac{W_0\left(\omega_n\frac{\xi}{L}\right)}{\int_{R_1}^{R_2}\xi^iW_1^2\left(\omega_n\frac{\xi}{L}\right)d\xi}\times$$
$$\times\sum_{\nu=1}^{3}\frac{\exp(s_\nu\tau)}{\Delta'(s_\nu)}\Big\{\mu_n^2\delta\overline{\Theta}+\Big[\tau_{rm}\frac{\varepsilon\rho}{c}s_\nu^2+$$

$$+\tau_{rm}\left(1+\frac{\varepsilon\rho}{c}\frac{\mu_n^2}{\mathrm{Lu}}\right)s_\nu-\mu_n^2\left(\mathrm{Fe}-\tau_{rm}\frac{1}{\mathrm{Lu}}\right)\Big]\bar{\varphi}+$$

$$+\tau_{rm}\left(s_\nu+\frac{\mu_n^2}{\mathrm{Lu}}\right)\bar{\psi}+\frac{\mu_n^2\delta}{c\gamma_0}F(\tau,\ q)-$$

$$-\frac{1}{\gamma_0}\left(s_\nu+\frac{\mu_n^2}{\mathrm{Lu}}\right)F(\tau,\ m)\Big\}.$$

The symbols introduced here are obvious from the foregoing.

We particularize the obtained solutions.

1. Plate

$$(i=0,\quad \xi=x,\quad R_1=0,\quad R_2=R=L).$$

The eigenfunction of the problem is

$$W_0(\omega_n x/R)=\cos\omega_n x/R.$$

The characteristic equation is

$$\sin\omega_n=0,\quad \omega_n=n\pi,\quad (n=1,\ 2,\ \ldots,\ \infty).$$

The function

$$W_1(\omega_n x/R)=\sin\omega_n x/R.$$

The integral

$$\int_0^R \sin^2\left(n\pi\frac{x}{R}\right)dx=\frac{1}{2}R.$$

2. Hollow cylinder

($i = 1$, $\xi = r$, R_1 is the radius of the internal surface, R_2 is the radius of the external surface, $L = R_1$).

The eigenfunction of the problem is

$$W_0(\omega_n r/R_1)=Y_1(\omega_n)J_0(\omega_n r/R_1)-J_1(\omega_n)Y_0(\omega_n r/R_1).$$

The characteristic equation is

$$W_1(k\omega_n)=Y_1(\omega_n)J_1(k\omega_n)-J_1(\omega_n)Y_1(k\omega_n)=0,$$

where

$$k=R_2/R_1.$$

The function

$$W_1(\omega_n r/R_1)=Y_1(\omega_n)J_1(\omega_n r/R_1)-J_1(\omega_n)Y_1(\omega_n r/R_1).$$

The integral

$$\int_{R_1}^{R_2} rW_1^2\left(\omega_n\frac{r}{R_1}\right)dr=\frac{2R_1^2}{\pi^2\omega_n^2}\frac{[J_1^2(\omega_n)-J_1^2(k\omega_n)]}{J_1^2(k\omega_n)}.$$

3. Solid cylinder

$$(i=1,\quad \xi=r,\quad R_1=0,\quad R_2=R=L).$$

The eigenfunction of the problem is

$$W_0(\omega_n r/R)=J_0(\omega_n r/R).$$

The characteristic equation is

$$J_1(\omega_n)=0.$$

The function

$$W_1(\omega_n r/R)=J_1(\omega_n r/R).$$

The integral

$$\int_0^R rJ_1^2\left(\omega_n\frac{r}{R}\right)dr=\frac{R^2}{2}J_0^2(\omega_n).$$

The second boundary-value problem of heat and moisture transfer with $\tau_{rm} = 0$ has been solved for a solid cylinder and sphere by Prudnikov [5] and for a hollow cylinder by Plyat [6].

NOTATION

T is the temperature; U is the moisture content; τ is the time; ξ is the coordinate; a is the thermal diffusivity; c is the specific heat; γ_0 is the density of absolutely dry body; a_m is the coefficient of moisture diffusion in body; ρ is the specific heat of phase transition; ε is the phase transition number; δ is the thermogradient coefficient, equal to ratio of thermodiffusion coefficient to coefficient of moisture diffusion in body; τ_{rm} is the moisture transfer relaxation period; R_j ($j = 1, 2$) is the coordinates of surfaces bounding body; L is the characteristic dimension of body; m_j is the rate of evaporation of moisture from surface; q_j is heat flux on surface; $W_0(\omega_n\xi/L)$ is the eigenfunction of problem; ω_n is the root of characteristic equation; $\mu_n^2=a_m\omega_n^2/L^2$; $\mathrm{Lu}=a_m/a$; $\mathrm{Fe}=\varepsilon\rho\delta/c$; $W_1\left(\omega_n\frac{\xi}{L}\right)=-\frac{\omega_n}{L}\left[W_0\left(\omega_n\frac{\xi}{L}\right)\right]'$.

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20 June 1966

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